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1

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TITLE LOBE AREA IN ADIABATIC HAMILTONIAN SYSTEMS

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LOBE AREA IN ADIABATIC HAMILTONIAN SYSTEMS

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Abstract We establish an analytically computable formula, based on the adiabatic Melnikov function, for lobe area in one-degree-of-freedom Hamiltonian systems depending on a parameter which varies slowly in time. We illustrate this lobe area result on a slowly parametrically forced pendulum, a paradigm problem for adiabatic chaos. Our analysis unites the theory of action from classical mechanics with the theory of the adiabatic Melnikov function from the field of global bifurcation theory.

Keywords: action, lobe area, adiabatic chaos, adiabatic Melnikov function, parametrically forced pendulum.

1 Introduction.

Planar Hamiltonian systems which depend on a parameter which varies slowly in time arise in the context of many physical problems. The equations of motion are:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(p,q,z) \\ \dot{p} &= -\frac{\partial H}{\partial q}(p,q,z) \\ \dot{z} &= \epsilon, \end{aligned} \tag{1}$$

where z is the parameter which varies slowly in time and H is the Hamiltonian. When $\epsilon = 0$, (1) is a one-parameter family of planar oscillators, and we shall refer to (1) as the frozen in time system.

When $0 < \epsilon \ll 1$, there are two qualitatively different regions on any instantaneous planar $p-q$ slice of the extended $p-q-t$ phase space of (1): a regular region and a separat ...-swept region. The regular region is defined as that area in which the adiabatic invariant (which is given to leading order by the action of an orbit, see [1]) is conserved. In this region the frequency of the unperturbed orbits is bounded away from zero for all values of the parameter. Furthermore, in the special case that the Hamiltonian depends periodically on z, the extension of KAM theory given in [2] guarantees that most periodic orbits of the planar systems which are sufficiently far away from any frozen separatrices survive as invariant tori on which the the flow is quasiperiodic when the parameter is allowed to vary slowly in time.

The complement of the regular region is the so-called separatrix-swept region. This region is defined as that area in which the unperturbed motion has a zero frequency for some value of the parameter. The separatrix-swept region has recently received a lot of attention. See for example [3], [4], [5], [6], [7], and [8]. The phenomenon of separatrix-crossing occurs in this $\mathcal{O}(1)$-sized region [4], orbits may evolve chaotically in the sense of the Smale-Birkhoff Homoclinic Theorem if H depends periodically on z [6], and the structure of the $\mathcal{O}(1)$ "chaotic sea" in periodic problems is very rich [7].

The organizing structures in the separatrix-swept region are the stable and unstable manifolds of hyperbolic orbits. The case which is of interest is when these manifolds intersect each other. For example, if the hyperbolic orbit is periodic or quasiperiodic in time, then its stable and unstable manifolds intersect infinitely many times and form a homoclinic (or heteroclinic) tangle. Problems in which there are only a finite number of intersections, *i.e.* in which the hyperbolic orbit has more general time-dependence, also arise, see for example [4], [5], and [10].

When there are two intersections of these stable and unstable manifolds in the $p-q$ plane at any instant of time, then they define a lobe in that plane. Knowing the area of a lobe analytically is important for our understanding of the separatrix swept region and also for many applications. In this paper, we establish a general exact lobe area formula as well as a closed form approximation, which is derived for arbitrary lobe shape, for (1) and discuss some of its physical consequences.

We first discuss the special case in which the Hamiltonian depends periodically on the slowly-varying parameter in Section 2. We relate our results to those obtained in [7]. Then in Section 3 we discuss the general case, in which H can have quite general dependence on z. We conclude this paper with some observations about which homoclinic orbits correspond to minima of the action and which correspond to minimaxs.

2 z-Periodic Hamiltonians.

In this section we establish analytically the observation that, when H depends periodically on z, the area of a lobe in (1) is asymptotic (as $\epsilon \to 0$) to A_s, which is the area in the $p-q$ plane between the two separatrices of the frozen in time system which enclose the maximum and minimum areas.

2.1 Asymptotic Lobe Area.

We assume that the frozen in time system has a curve of hyperbolic fixed points, which may be written as a graph over z, $\gamma(z)$, each of which has an orbit homoclinic to it. We denote the homoclinic orbit on the instantaneous $p-q$ plane of the extended $p-q-z$ phase space of the frozen in time system by Γ^z. This assumption is made with out loss of generality, for if (1) has more than one homoclinic orbit for a given z than our results apply to each one individually, and if (1) has a heteroclinic orbit then the results apply to it as well. From the persistence theory of hyperbolic invariant manifolds, we know that this curve of fixed points (an invariant manifold) becomes a hyperbolic periodic orbit, γ_ϵ, when $\epsilon \neq 0$. We also know that γ_ϵ has two-dimensional stable and unstable manifolds (see [11]), denoted by $W^S(\gamma_\epsilon)$ and $W^U(\gamma_\epsilon)$ and on which orbits approach and leave γ_ϵ, respectively, exponentially. Furthermore, from the theory of the adiabatic Melnikov function [9], we know that if the adiabatic Melnikov function is periodic in z and has an infinite set of simple zeroes, then one branch of each of $W^S(\gamma_\epsilon)$ and $W^U(\gamma_\epsilon)$ intersect each other along infinitely many curves in which the orbits homoclinic to γ_ϵ lie (see Figure 1). The points at which these homoclinic orbits intersect the Poincaré surface of section are called primary intersection points [12].

Let $\mathcal{P}$ and $\mathcal{Q}$ be the orbits of the two adjacent primary intersection points P and Q which define the lobe L^{PQ} on the instantaneous $p-q$ plane at time time t_2, denoted by Π_{t_2}. The area of L^{PQ} is given exactly by the difference in the actions of $\mathcal{P}$ and $\mathcal{Q}$ [13]:

$$A = \int_{\mathcal{P}-\mathcal{Q}} pdq - Hdt. \tag{2}$$

Since there does not appear to be a way to compute A in closed form from (2), we give an approximate expression which may be evaluated analytically in two ways and then show that the error made in this approximation vanishes asymptotically at least as fast as ϵ when $\epsilon \to 0$.

Let Γ^{Z_0} be the separatrix of the frozen in time system which encloses the maximum area and Γ^{Z_1} be the separatrix which encloses the minimum area. The approximate lobe area is given by the flux of the perturbed vector field through the piece of the unperturbed homoclinic manifold with z restricted to lie in the interval $[Z_0, Z_1]$, see Figure 2:

$$A_0 = \int\int_\Sigma dp \wedge dq - dH \wedge dt. \tag{3}$$

This integral can be evaluated in closed form because the integrand is evaluated on the known solutions lying on Σ. The surface integral in (3) may be written as a line integral around the boundary of Σ by Stokes' Theorem:

$$A_0 = \int_{\Gamma^{z_0}} (pdq - Hdt) - \int_{\Gamma^{z_1}} (pdq - Hdt). \tag{4}$$

Then, observing that $dt = 0$ on the instantaneous $p - q$ plane Π_{t_2}, and hence $dt = 0$ also along Γ^{Z_0} and Γ^{Z_1}, we arrive at

$$A_0 = \int_{\Gamma^{z_0}} pdq - \int_{\Gamma^{z_1}} pdq \equiv A_s. \tag{5}$$

A second way in which (4) may be evaluated analytically is in terms of the adiabatic Melnikov function $M_A(z)$, which is defined as

$$M_A(z) = \int_{-\infty}^{\infty} \left(\frac{\partial H}{\partial z}(p_0^z(s), q_0^z(s), z) - \frac{\partial H}{\partial z}(\gamma(z), z) \right) ds, \tag{6}$$

where s is the time variable of the frozen system, and $(p_0^z(s), q_0^z(s), z)$ is the solution of the frozen in time system on Γ^z, see [14] and [6]. A computationally more convenient form of $M_A(z)$ can be derived from the definition using integration by parts on s and realizing that the boundary terms vanish:

$$M_A(z) = -\int_{-\infty}^{\infty} s \frac{d}{ds} \frac{\partial H}{\partial z}(p_0^z(s), q_0^z(s), z) = \int_{-\infty}^{\infty} s\{H, \frac{\partial H}{\partial z}\}(p_0^z(s), q_0^z(s), z), \tag{7}$$

where $\{H, \frac{\partial H}{\partial z}\}$ denotes the Poisson Bracket of H and $\frac{\partial H}{\partial z}$ with respect to (p, q). We remark that the integrand maybe written as $\{H, \frac{\partial H}{\partial t}\}$.

In [8], we perform this second evaluation of A_0, by changing coordinates from the pair (p, q) and from the pair (H, t) to the pair (s, z), which are good coordinates for the homoclinic manifold. We obtain:

$$A_0 = \int_{Z_0}^{Z_1} \int_{-\infty}^{\infty} \left(-\frac{dH}{dz} + \frac{\partial H}{\partial z} \right) (p_0^z(s), q_0^z(s), z) ds dz. \tag{8}$$

Now, we add $\frac{dH}{dz}(\gamma(z), z)$ and subtract $\frac{\partial H}{\partial z}(\gamma(z), z)$, which are equal since the other partial derivatives at the hyperbolic fixed point vanish, from the integrand and use the fact that $\frac{dH}{dz}(p_0^z(s), q_0^z(s), z) = \frac{dH}{dz}(\gamma(z), z)$ for all s to arrive at:

$$A_0 = \int_{Z_0}^{Z_1} M_A(z) dz. \tag{9}$$

Finally, we show that the error made in the approximation vanishes at least as fast as ϵ. We do this by evaluating the difference between the exact and the approximate lobe areas.

Geometrically, this difference is given by the flux of (1) through any two surfaces which connect Σ and $W^U(\gamma_\epsilon)$ when $z \leq z_2 = \epsilon t_2$ and $W^S(\gamma_\epsilon)$ when $z \geq z_2$. This can be seen from the following picture: Together with Σ and pieces of the stable and unstable manifolds of the hyperbolic periodic orbit ($W^S(\gamma_\epsilon)$ when $z \leq z_2$ and $W^U(\gamma_\epsilon)$ when $z \geq z_2$ where $z_2 = \epsilon t_2$ and we are looking at the instantaneous plane Π_{t_2}), the two connecting surfaces form an infinitely long tube whose cross-section in the $p - q$ plane is four-sided when $z \in [Z_0, Z_1]$ and three-sided for z outside this interval with a hole in its side on the reference plane Π_{t_2} which is exactly the lobe L^{PQ}. We label the connecting surfaces by Ω^P and Ω^Q in Figure 3. In [8] we show that, as $z \to \pm\infty$, the cross-sectional area of the tube vanishes exponentially. Therefore, the net flux through all of the sides of the tube is equal to the area of the "hole", *i.e.* the area of the lobe L^{PQ}. Now, since there is no flux through eithe. $W^S(\gamma_\epsilon)$ or $W^U(\gamma_\epsilon)$, the area of the "hole" is equal to flux through Σ, which is given by our approximation A_0, plus the flux through the two connecting surfaces. Thus, it remains only to determine the flux through the connecting surfaces Ω^P and Ω^Q. In [8], we prove that this flux, *i.e.*, the error in our approximation, is $\mathcal{O}(\epsilon)$.

Thus the numerical observation about the asymptotic behavior of the lobe area in (1) when H depends periodically in z is established analytically. As the area enclosed by the frozen separatrices Γ^z increases, the region inside the homoclinic tangle ingests phase space area, and vice versa, as the area enclosed decreases, the inner region ejects phase space area. Hence, during the z-interval in which the area enclosed by the frozen separatrices increases from its minimum value to its maximum value, an area approximately equal to A_s enters the region inside the homoclinic tangle. Similarly, during the z-interval in which the area enclosed by the frozen separatrices decreases from its maximum value to its minimum value, an area approximately equal to A_s leaves the region inside the homoclinic tangle. Thus, we expect the lobe area to be given by A_s to leading order since phase space area enters and exits the region inside the homoclinic tangle only through the turnstile lobes.

2.2 The Adiabatic Pendulum.

The results of the above theorem is nicely illustrated on the pendulum with slowly varying base support, which is governed by the Hamiltonian

$$H = \frac{p^2}{2} - (1 - \gamma \cos z)\cos q, \tag{10}$$

where $\dot{z} = \epsilon$ and $\gamma \in (0,1)$. For every value of $z \in [0, 2\pi)$, the autonomous frozen in time system system has hyperbolic fixed points at $(k\pi, 0)$, for all $k \in \mathbf{Z}$. We look at $(-\pi, 0)$ and $(\pi, 0)$. These two points are connected to each other by upper and lower separatrices parametrized by

$$(p_0^z(s), q_0^z(s)) = (\pm 2\sqrt{1-\gamma\cos z}\,\mathrm{sech}(\sqrt{1-\gamma\cos z}\,s), \pm 2\arcsin(\tanh\sqrt{1-\gamma\cos z}\,s)),$$

where s is the time variable for (1) when $\epsilon = 0$. For convenience we consider only the upper half plane $p > 0$. The picture for the lower half plane is obtained by a 180^0

rotation. From the definition, we compute

$$
\begin{aligned}
M_A(z) &= \gamma \int_{-\infty}^{+\infty} s p_0^z(s) \sin(q_0^z(s)) \sin z \, ds \\
&= \frac{4\gamma \sin z}{\sqrt{1-\gamma\cos z}}. \qquad (11)
\end{aligned}
$$

Thus we see that M_A has an infinite number of simple zeroes, and we know that the stable and unstable manifolds of γ_ϵ intersect transversely in an infinite sequence of curves for ϵ sufficiently small. The Poincaré maps for $z_0 = 0, \gamma = 0.75$, and various values of ϵ are shown in Figure 5. We remark that the fact that the zeroes of M_A are an $\mathcal{O}(\frac{1}{\epsilon})$ time-of-flight apart implies that all, except for possibly one, of the pip's are exponentially close to γ_ϵ. We also remark that the three regular regions sufficiently far inside and outside the homoclinic tangle are filled with KAM tori.

From (11) we compute

$$
\begin{aligned}
A_0 &= 4\gamma \int_0^\pi \frac{\sin z}{\sqrt{1-\gamma\cos z}} dz \\
&= 8(\sqrt{1+\gamma} - \sqrt{1-\gamma}) \\
&= A_S \qquad (12)
\end{aligned}
$$

where A_S is precisely the difference in the areas enclosed by the separatrices of the unperturbed system in the upper half plane corresponding to $z = \pi$ and $z = 0$, respectively. See Figure 4. Finally, the general result demonstrated above that $A = A_S + \mathcal{O}(\epsilon)$ is what we observe numerically (see Table 1).

3 Hamiltonians with General z-Dependence.

The three results for z-periodic Hamiltonians also hold when H has quite general dependence on z. We assume that H is at least C^3 with respect to z. First, we know that a curve of uniformly hyperbolic fixed points which can be written as a graph over the z variable persists as a hyperbolic orbit with stable and unstable manifolds when $\epsilon \neq 0$. The second result which carries over to the general case is the exact lobe area formula. In [8], we give an alternative derivation of (2) which does not depend on the vector field having a recurrent section, as (1) has when H depends periodically on z. Finally, the main result about the asymptotic lobe area, established in the previous section, holds in the general case as well, and hence A_0 as given by either (3) or (4) is the lobe area to within an asymptotic error of $\mathcal{O}(\epsilon)$. In fact, the derivation of the result given in [8] does not depend on H being periodic in z, only that H is at least C^3 in z.

The main result represents a generalization of the existing techniques to measure the areas of lobes. The two main techniques to measure the lobe area developed to date apply only to two-dimensional maps or to vector fields which have a recurrent planar section. The first technique involves computing the action of the two adjacent intersection orbits which define the lobe [13]. However, this technique does not appear to lead to a closed form expression for the area. As stated above (2) can be derived for the general case and a closed-form approximation can be made. The second technique is a direct geometric measurement involving the regular Melnikov function [12]. This technique can only be used if the shape of the lobe is such that the pieces of the stable and unstable manifolds which define it can be written as graphs over the unperturbed separatrix. Hence, this direct technique cannot be used in adiabatic systems where the lobe shape does not satisfy this property, see Figure 5. As mentioned before, our results are independent of the shape of the lobe, and thus overcome the limitation inherent in the second technique.

3.1 A Maximal Property for Lobe Area.

The only observation which does not carry over directly from the special periodic case to the general case is that $A_0 = A_s$, because A_s is only defined for z-periodic H. Never the less, this observation does have a counterpart in the general case. We now establish a maximal property for the lobe area which applies to the general case and reduces to $A_0 = A_s$ in the special case.

In particular, we show that the zeroes of the adiabatic Melnikov function occur in such a way that the area A_0 (determined by the frozen separatrices Γ^{Z_0} and Γ^{Z_1}) is locally the largest possible. We take $\Sigma(z_i, z)$ to be the piece of Γ with the variable z restricted to the interval $[z_i, z]$, where z_i is a constant. Second, we define the function:

$$A(z_i, z) = \int\int_{\Sigma(z_i,z)} dp \wedge dq, \tag{13}$$

which is the area of the projection of $\Sigma(z_i, z)$ on a slice Π_z. By a calculation similar to that done in the previous section, the function A can also be written as:

$$A(z_i, z) = \int_{z_i}^{z} \Big[\int_{-\infty}^{\infty} s\{H, \frac{\partial H}{\partial z}\} ds\Big] dz, \tag{14}$$

which is the integral of $M_A(z)$ over $\Sigma(z_i, z)$. Now,

$$\frac{dA}{dz}(z_i, \bar{z}) = \int_{-\infty}^{\infty} s\{H, \frac{\partial H}{\partial z}\}\Big|_{z=\bar{z}} ds = M_A(\bar{z}). \tag{15}$$

Hence we know that Z_0 and Z_1 are simple zeroes of $\frac{dA}{dz}(z_i, \bar{z})$, because Z_0 and Z_1 are simple zeroes of $M_A(z)$. Next we remark that because $\frac{dM_A}{dz}(z)$ has different signs at Z_0 and Z_1,

$$\frac{d^2 A}{dz^2}(z_i, Z_0) \quad \text{and} \quad \frac{d^2 A}{dz^2}(z_i, Z_1) \tag{16}$$

are of opposite signs, as well. Thus the adiabatic Melnikov function picks out in a natural fashion a pair of adjacent local extremas, one minimum and one maximum, of $A(z_i, z)$ and the local extremal frozen separatrices Υ^{Z_0} (maximal) and Υ^{Z_1} (minimal), respectively. We remark that in the special case discussed in Section 2, we get $A(Z_1, Z_0) = A_s$, as stated there.

3.2 Action Minimizing and Minimax Homoclinic Orbits.

We conclude this section with the following remark. The orbits of (1) homoclinic to γ_ϵ are of two types. One corresponds to a minimum of the action, and the other type corresponds to a minimax (*i.e.*, local maximum) of the action, see [15]. The observation made above about M_A determines this correspondence.

As $\epsilon \to 0$, the homoclinic orbit $\mathcal{Q}$ limits on the separatrix Γ^{Z_1}. Therefore, because Γ^{Z_1} is a local minimum of $A(z_i, z)$, $\mathcal{Q}$ is a local minimum of the action. Similarly, as $\epsilon \to 0$, the homoclinic orbit $\mathcal{P}$ limits on the separatrix Γ^{Z_0}. Therefore, because Γ^{Z_0} is a local maximum of $A(z_i, z)$, $\mathcal{P}$ is a local maximum of the action. In other words, the homoclinic orbit corresponding to $\bar{z}$ such that $\frac{dM_A(\bar{z})}{dz} > 0$ and $M_A(\bar{z}) = 0$ is a local minimum, and the homoclinic orbit corresponding to $\bar{z}$ such that $\frac{dM_A(\bar{z})}{dz} < 0$ and $M_A(\bar{z}) = 0$ is a local maximum. We remark that a similar correspondence, between the well-known Melnikov function for small-amplitude perturbations (see [16]) and the type (either action-minimizing or minimax) of the homoclinic orbit, must also apply in the case of small amplitude perturbations where the geometric identification of the minimum and the minimax homoclinic orbits is not as clear as it is for adiabatic problems.

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Figure Captions

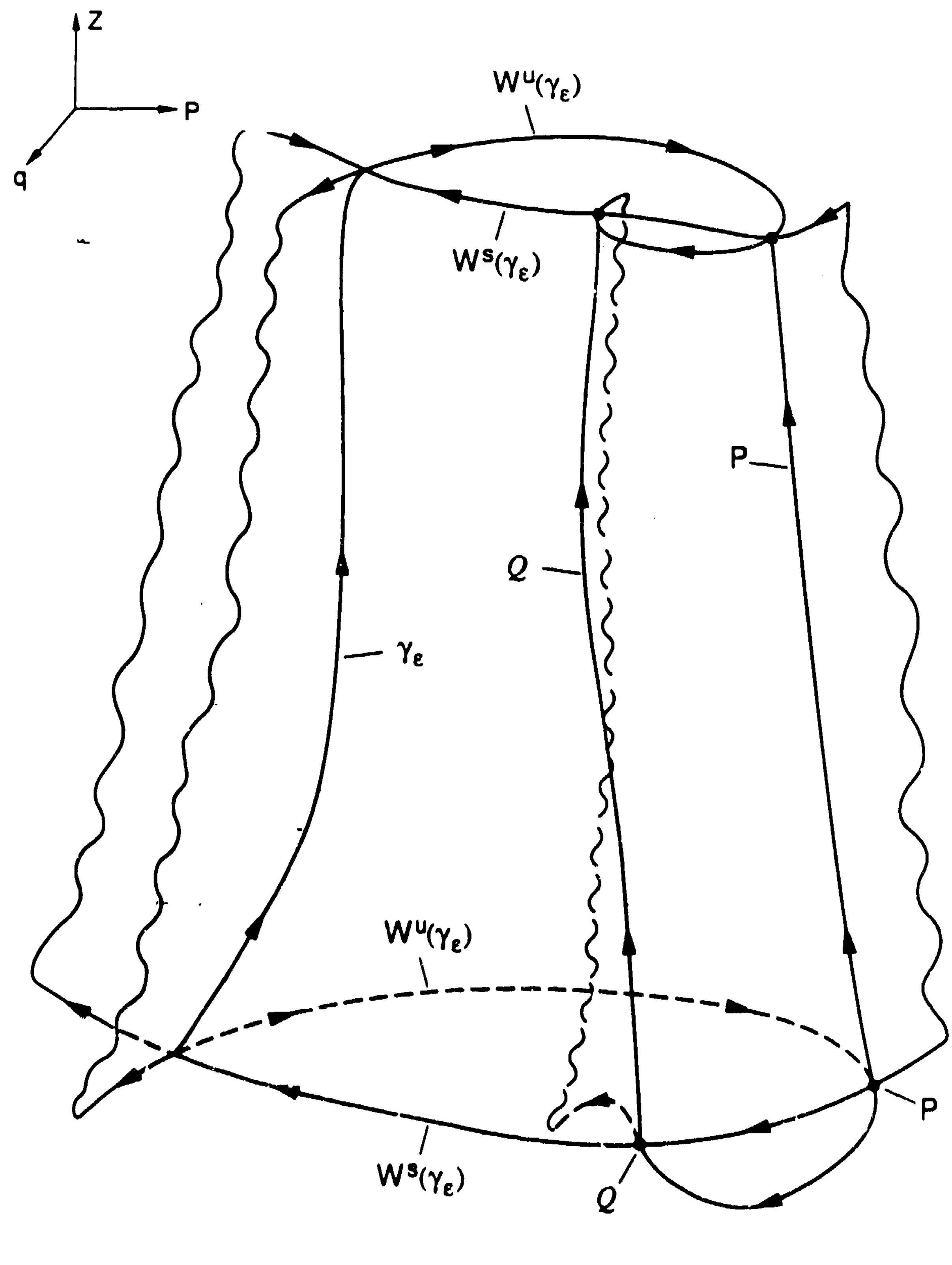
1. Geometry of (1) in the extended $p - q - z$ phase space with $0 < \epsilon \lll 1$.

2. The section, Σ, of the homoclinic manifold Γ used to get the approximation, A_0.

3. The error made in the approximation is given by the flux of the perturbed vector field through the connecting surfaces $\Omega^{\mathcal{P}}$ and $\Omega^{\mathcal{Q}}$.

4. Unperturbed pendulum phase portrait, with $\gamma = 0.7$.

5. Sequence of Poincaré maps which show that lobes become larger as ϵ gets smaller. The example is the adiabatic pendulum of Section 6 with $\gamma = 0.75$. a: $\epsilon = \frac{2\pi}{12}$. b: $\epsilon = \frac{2\pi}{20}$. c: $\epsilon = \frac{2\pi}{25}$.



Z
P
q
$W^u(\gamma_\varepsilon)$
$W^s(\gamma_\varepsilon)$
P
Q
γ_ε
$W^u(\gamma_\varepsilon)$
P
$W^s(\gamma_\varepsilon)$
Q

FIG. 1

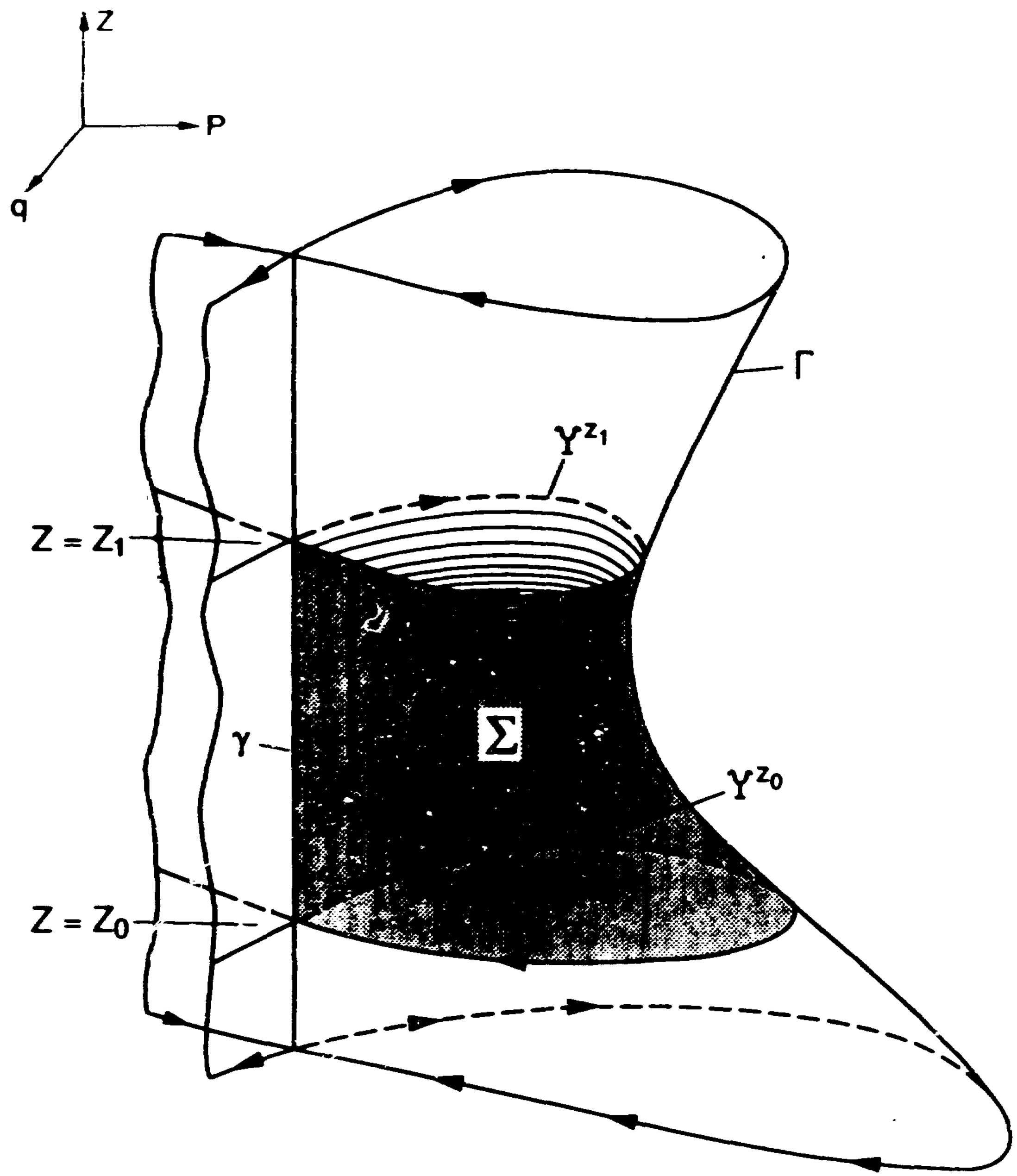


FIG. 2

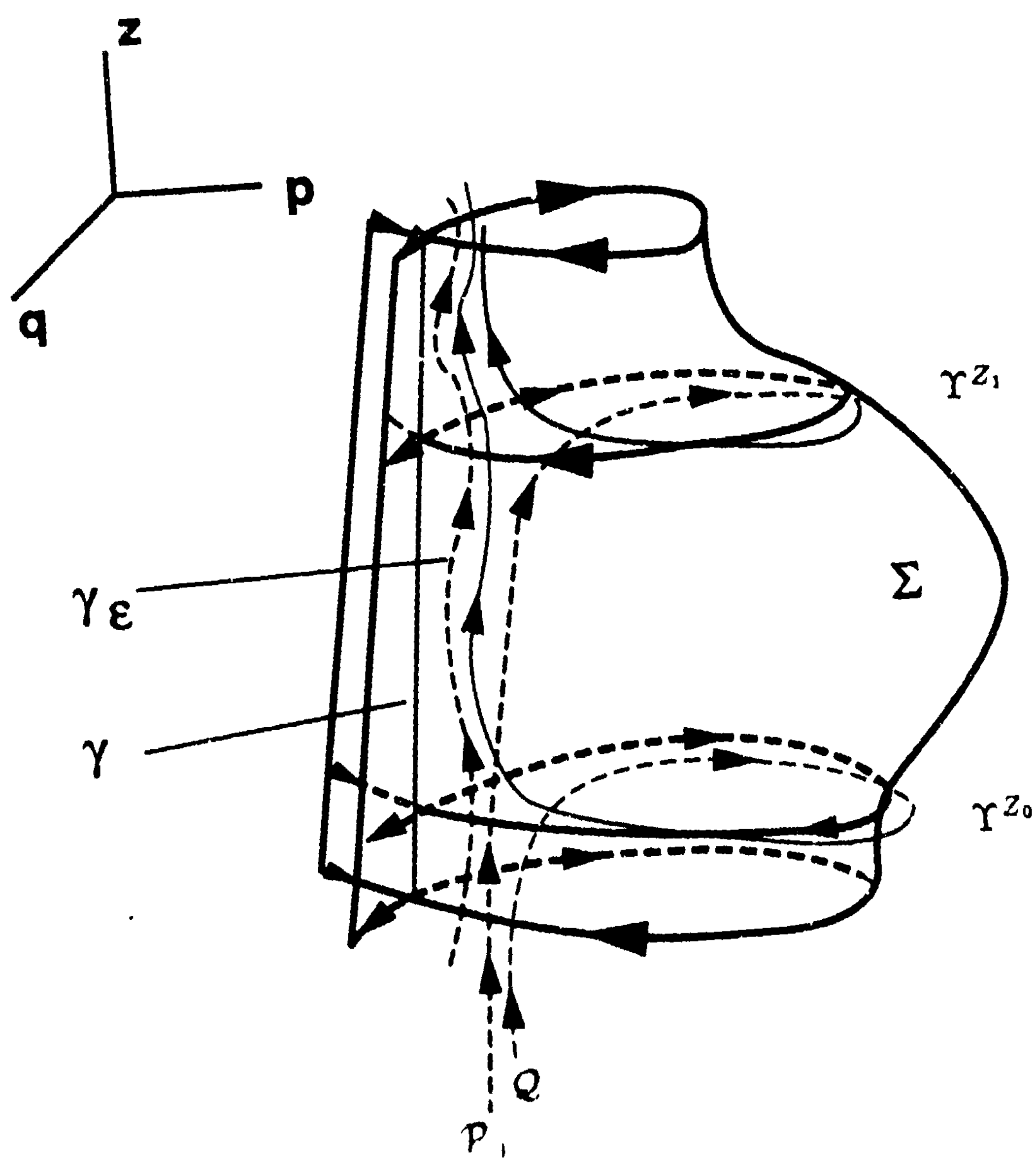


FIG. 3. A

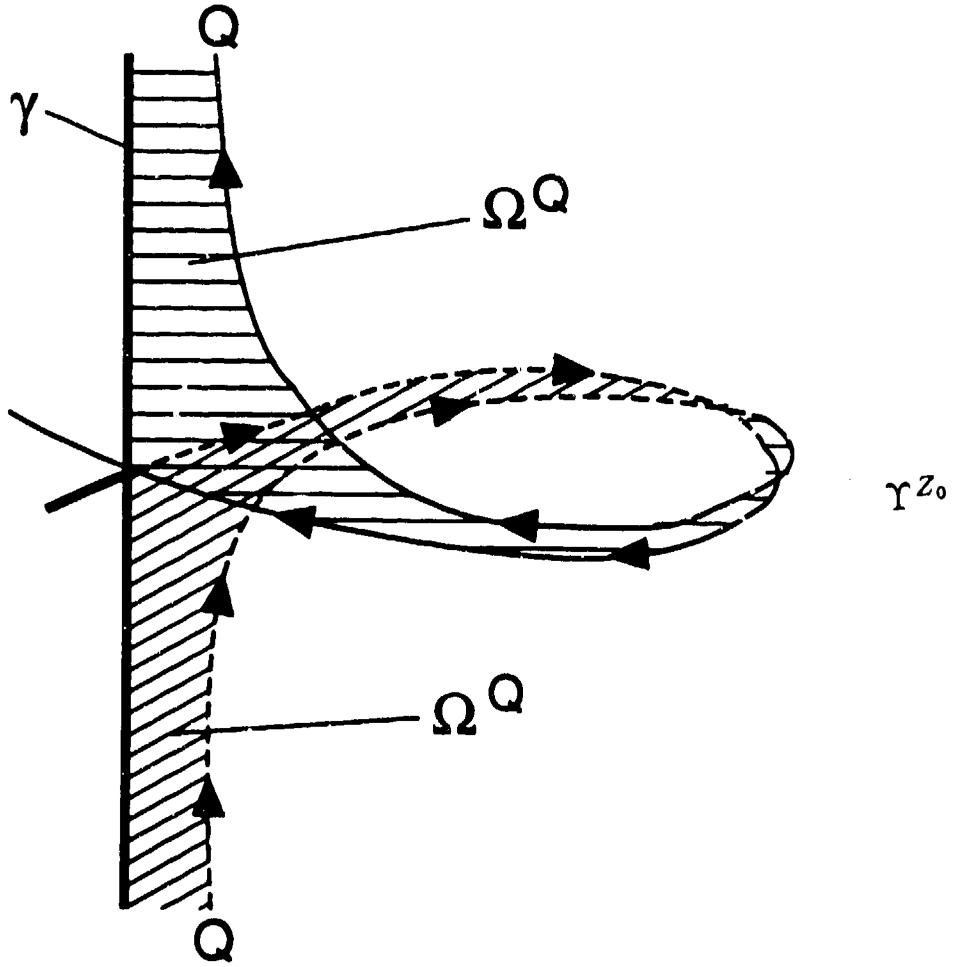
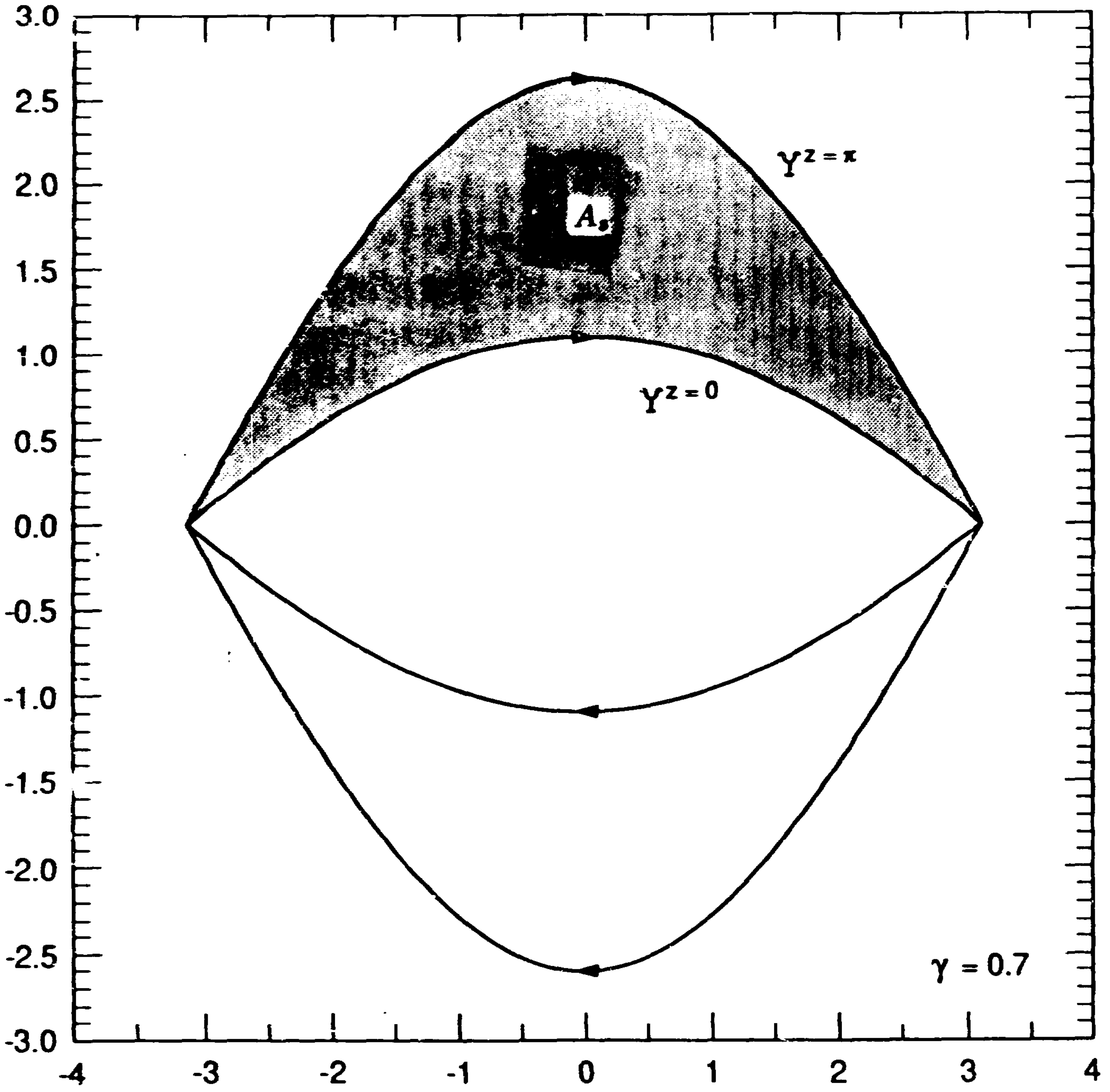


FIG 3.B



3.0
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
-3.0
-4
-3
-2
-1
0
1
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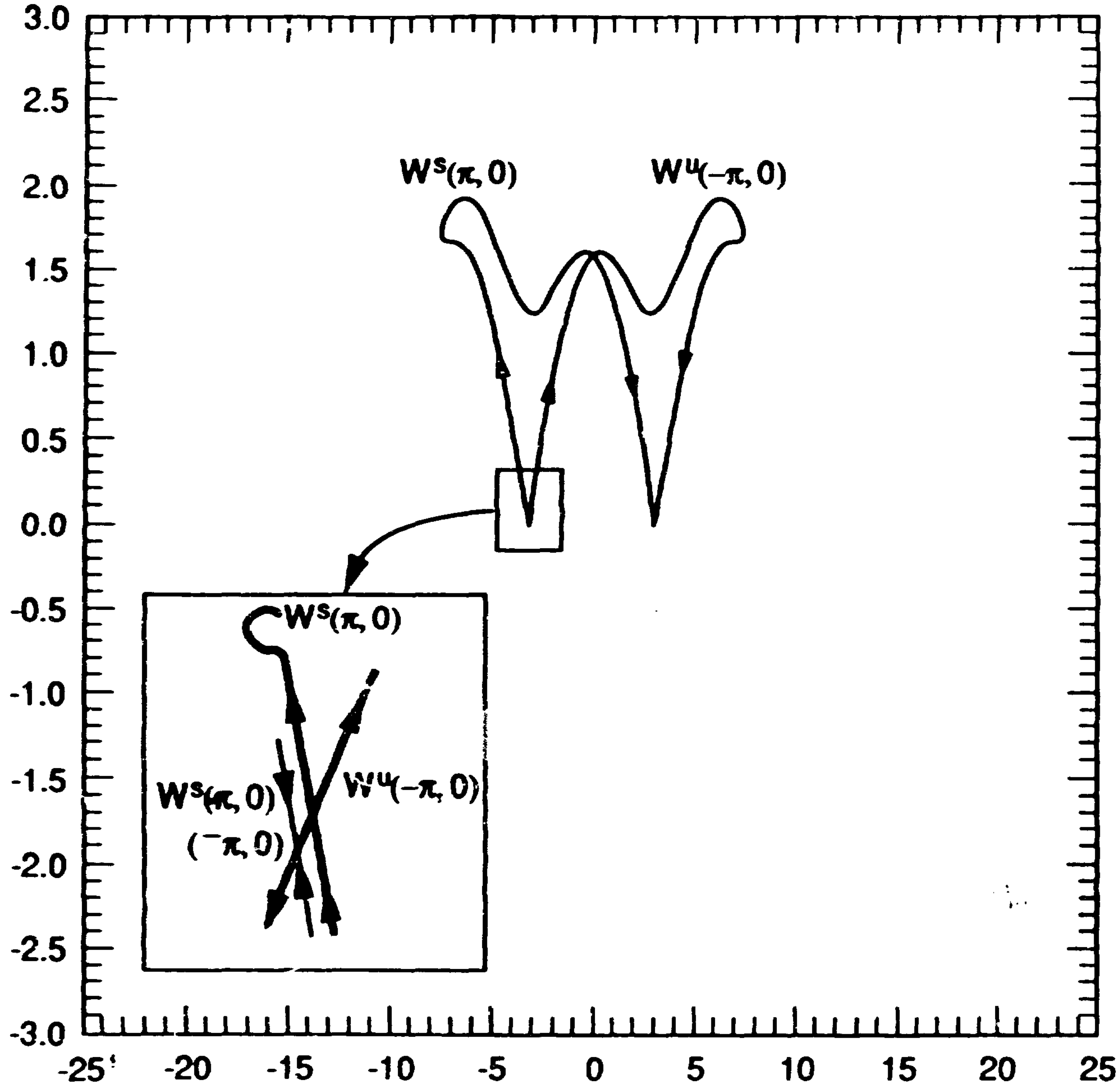
3
4
$Y^{z=\pi}$
A_s
$Y^{z=0}$
$\gamma = 0.7$

FIG. 4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-3.0
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-15
-10
-5
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5
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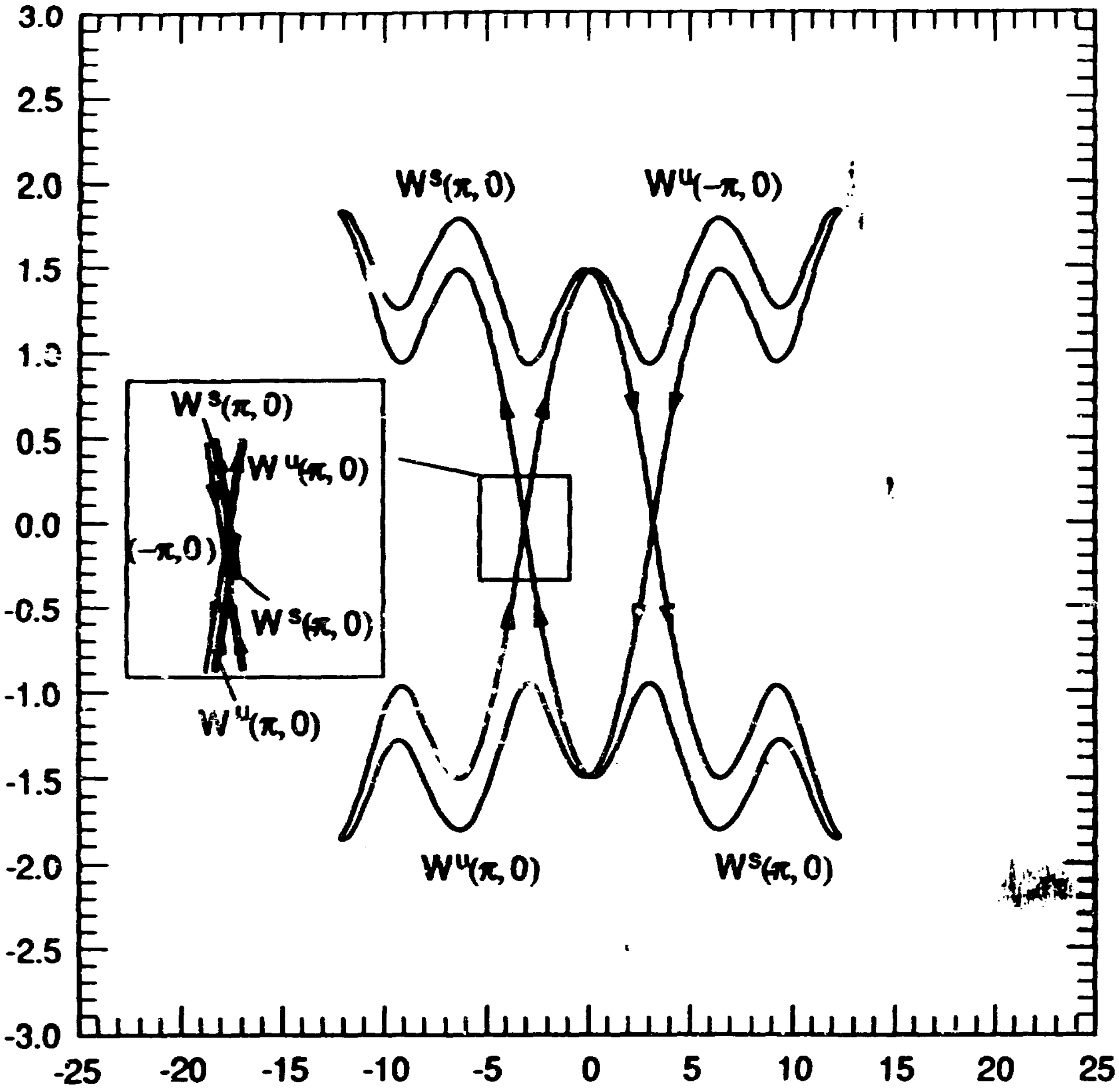
20
25
Wˢ(π, 0)
Wᵘ(−π, 0)
Wˢ(π, 0)
Wˢ(π, 0)
(⁻π, 0)
Wᵘ(−π, 0)

FIG. 5.A.



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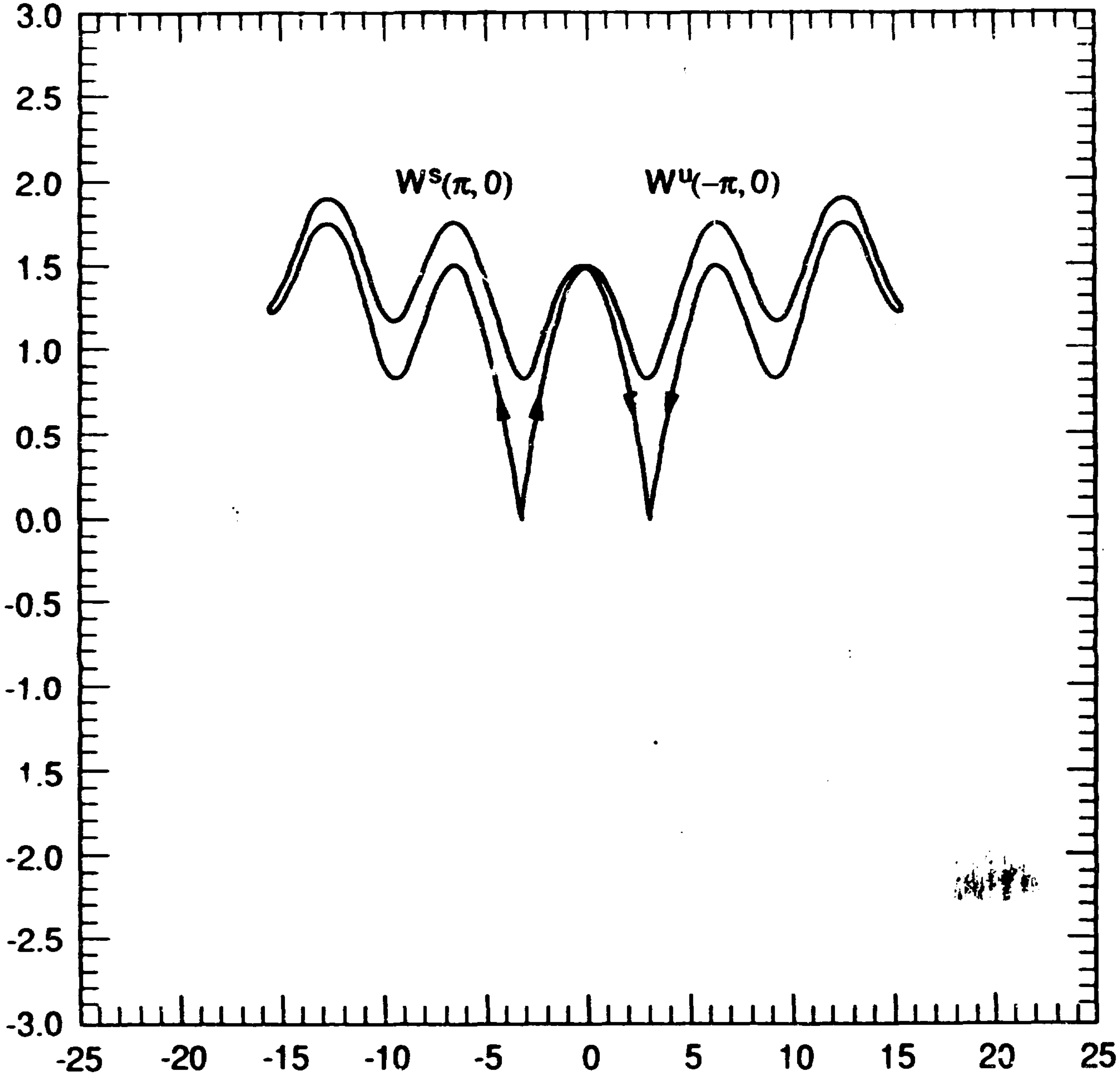
-10
-5
0
5
10
15
20
25
$W^s(\pi, 0)$
$W^u(-\pi, 0)$
$W^s(\pi, 0)$
$W^u(\pi, 0)$
$(-\pi, 0)$
$W^s(\pi, 0)$
$W^u(\pi, 0)$
$W^u(\pi, 0)$
$W^s(\pi, 0)$

FIG. 5. B.



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-3.0
-25
-20
-15
-10
-5
0
5
10
15
20
25
$W^s(\pi, 0)$
$W^u(-\pi, 0)$

FIG. 5.C.

	epsilon	lobe area
$\gamma = 0.5$	$2\pi/15$	3.74
$\gamma = 0.5$	$2\pi/18$	3.84
$\gamma = 0.5$	$2\pi/20$	3.88
$\gamma = 0.5$	$2\pi/25$	3.94
$\gamma = 0.5$	$2\pi/30$	3.97
$\gamma = 0.5$	0	$A_0 = A_s = 4.14$

Table 1: Lobe area vs. ϵ based on a numerical trapezoidal rule sum integration. The lobe area increases as ϵ decreases. Numerical solution of $(5.1)_\epsilon$ done using a fourth order symplectic integrator.